

## ON SOME GENERALIZATIONS OF BCC-ALGEBRAS

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ABSTRACT. We describe weak BCC-algebras (also called BZ-algebras) in which the condition  $(xy)z = (xz)y$  is satisfied only in the case when elements  $x, y$  belong to the same branch. We also characterize branchwise commutative and branchwise implicative weak BCC-algebras satisfying this condition. We also describe connections between various types of implicative weak BCC-algebras.

## 1. INTRODUCTION

Realistic simulation of human decision making process has been a goal for artificial intelligence development for decades. Decisions that are made based on both certain and uncertain types of information are a special focus and the logic behind those decisions is dominant in proof theory. Such logic is at the core of any system or a tool for applications, in both mathematics and computers. In addition to the classical logic, many logic systems that deal with various aspects of uncertainty of information (e.g. fuzziness, randomness, etc.) are built on top of it, such as many-valued logic or fuzzy logic. A real life example of such uncertainty may be incomparability of data. To deal with fuzzy and uncertain information, computer science relies heavily on non-classical logic.

In recent years, motivated by both theory and application, the study of  $t$ -norm-based logic systems and the corresponding pseudo-logic systems has become a great focus in the field of logic. Here,  $t$ -norm-base algebraic investigations were first to the corresponding algebraic investigations, and in the case of pseudo-logic systems, algebraic development was first to the corresponding logical development (see for example [17]). It is well known that BCK and BCI-algebras are inspired by some implicational logic. This inspiration is illustrated by the similarities of names. We have BCK-algebras and a BCK positive logic, BCI-algebras and a BCI positive logic and so on. In many cases, the connection between such algebras and their corresponding logics is much stronger. In such cases one can give a translation procedure which translates all well formed formulas and all theorems of a given logic  $\mathcal{L}$  into terms and theorems of the corresponding algebra. Nevertheless the study of algebras motivated by known logics is interesting and very useful for corresponding logics also in the case when the full inverse translation procedure is impossible.

To solve some problems on BCK-algebras Y.Komori introduced in [21] the new class of algebras called BCC-algebras. In view of strongly connections with a  $\text{BIK}^+$ -logic, BCC-algebras also are called  $\text{BIK}^+$ -algebras (cf. [24]) or BZ-algebras (cf. [25]). Nowadays, the mathematicians especially from China, Japan and Korea, have been studying various generalizations of BCC-algebras. All these algebras

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2010 *Mathematics Subject Classification.* 03G25, 06F35

*Key words and phrases.* weak BCC-algebra, BCC-algebra, branchwise commutative, branchwise implicative weak BCC-algebra, BZ-algebra, branch, condition (S).

have one distinguished element and satisfy some common identities. One of very important identities is the identity  $(xy)z = (xz)y$ . This identity is satisfied in such algebras as pre-logics (cf. [1]), Hilbert algebras and implication algebras (cf. [2]) strongly connected with MV-algebras (cf. [4]). This identity also holds in BCK-algebras and some their generalizations, but not in BCC-algebras. BCC-algebras satisfying this identity are BCK-algebras (cf. [8] or [9]). The class of all bounded commutative BCC-algebras is equivalent to the class of all MV-algebras (cf. [5]).

Therefore, it makes sense to consider such BCC-algebras and some their generalizations for which this identity is satisfied only by elements belonging to some subsets (cf. [11]).

On the other hand, many mathematicians investigate BCI-algebras in which some basic properties are restricted to some subset called branches. For weak BCC-algebras such study was initiated in [11] and continued in [20].

In this paper we describe branchwise commutative and branchwise implicative weak BCC-algebras in which the condition  $(xy)z = (xz)y$  is satisfied only in the case when elements  $x, y$  belong to the same branch. We also characterize branchwise commutative and branchwise implicative weak BCC-algebras satisfying this condition. We also describe connections between various generalizations of implicative weak BCC-algebras. Finally, we consider weak BCC-algebras with condition (S).

## 2. BASIC DEFINITIONS AND FACTS

The BCC-operation will be denoted by juxtaposition. Dots will be used only to avoid repetitions of brackets. For example, the formula  $((xy)(zy))(xz) = 0$  will be written in the abbreviated form as  $(xy \cdot zy) \cdot xz = 0$ .

**Definition 2.1.** A *weak BCC-algebra* is a system  $(G; \cdot, 0)$  of type  $(2, 0)$  satisfying the following axioms:

- (i)  $(xy \cdot zy) \cdot xz = 0$ ,
- (ii)  $xx = 0$ ,
- (iii)  $x0 = x$ ,
- (iv)  $xy = yx = 0 \implies x = y$ .

By many mathematicians, especially from China and Korea, weak BCC-algebras are called *BZ-algebras* (see for example [25]), but we save the first name because it coincides with the general concept of names used for algebras inspired by various logics.

A weak BCC-algebra satisfying the identity

$$(v) \quad 0x = 0$$

is called a *BCC-algebra*. A BCC-algebra with the condition

$$(vi) \quad (x \cdot xy)y = 0$$

is called a *BCK-algebra*.

An algebra  $(G; \cdot, 0)$  of type  $(2, 0)$  satisfying the axioms (i), (ii), (iii), (iv) and (vi) is called a *BCI-algebra*. A weak BCC-algebra is a BCI-algebra if and only if it satisfies the identity  $xy \cdot z = xz \cdot y$  (cf. [8]).

A weak BCC-algebra which is not a BCC-algebra is called *proper* if it is not a BCI-algebra. A proper weak BCC-algebra has at least four elements. But there are only two such non-isomorphic weak BCC-algebras (see [10]).

In any weak BCC-algebra we can define a natural partial order  $\leq$  putting

$$(1) \quad x \leq y \iff xy = 0.$$

This means that a weak BCC-algebra can be considered as a partially ordered set with some additional properties.

**Proposition 2.2.** *An algebra  $(G; \cdot, 0)$  of type  $(2, 0)$  with a relation  $\leq$  defined by (1) is a weak BCC-algebra if and only if for all  $x, y, z \in G$  the following conditions are satisfied:*

$$(i') \quad xy \cdot zy \leq xz,$$

$$(ii') \quad x \leq x,$$

$$(iii') \quad x0 = x,$$

$$(iv') \quad x \leq y \text{ and } y \leq x \text{ imply } x = y. \quad \square$$

From (i') it follows that in weak BCC-algebras implications

$$(2) \quad x \leq y \implies xz \leq yz$$

$$(3) \quad x \leq y \implies zy \leq zx$$

are valid for all  $x, y, z \in G$ .

In the investigations of algebras connected with various types of logics an important role plays the self map  $\varphi(x) = 0x$ . This map was formally introduced in [13] for BCH-algebras, but earlier it was used in [6] and [7] to investigate some classes of BCI-algebras. Later, in [14], it was used to characterize some ideals of weak BCC-algebras. Recall that a subset  $A$  of a weak BCC-algebra is called a *BCC-ideal* if  $0 \in A$ , and for all  $y \in A$  from  $xy \cdot z \in A$  it follows  $xz \in A$ . A special case of a BCC-ideal is a *BCK-ideal*, i.e., a subset  $A$  such that  $0 \in A$ , and  $y, xy \in A$  imply  $x \in A$ . In the literature BCK-ideals also are called *ideals*.

The main properties of this map are collected in the following theorem proved in [14].

**Theorem 2.3.** *Let  $G$  be a weak BCC-algebra. Then*

$$(1) \quad \varphi^2(x) \leq x,$$

$$(2) \quad x \leq y \implies \varphi(x) = \varphi(y),$$

$$(3) \quad \varphi^3(x) = \varphi(x),$$

$$(4) \quad \varphi^2(xy) = \varphi^2(x)\varphi^2(y)$$

for all  $x, y \in G$ .  $\square$

A weak BCC-algebra in which  $\text{Ker}\varphi(x) = \{0\}$  is called *group-like* or *anti-grouped*. A weak BCC-algebra  $(G; \cdot, 0)$  is group-like if and only if there exists a group  $(G; *, 0)$  such that  $xy = x * y^{-1}$  (cf. [7], [12] or [25]).

The set of all minimal (with respect to  $\leq$ ) elements of  $G$  will be denoted by  $I(G)$ . It is a subalgebra of  $G$ . Moreover,

$$I(G) = \varphi(G) = \{a \in G : \varphi^2(a) = a\}$$

(cf. [12]). The set

$$B(a) = \{x \in G : a \leq x\},$$

where  $a \in I(G)$  is called the *branch* initiated by  $a$ . Branches initiated by different elements are disjoint (cf. [14]). Comparable elements are in the same branch, but there are weak BCC-algebras containing branches in which not all elements are comparable.

**Lemma 2.4.** (cf. [12]) *Elements  $x$  and  $y$  are in the same branch if and only if  $xy \in B(0)$ .*  $\square$

**Lemma 2.5.** (cf. [12])  *$B(0)$  is a subalgebra of  $G$ . It is a maximal BCC-algebra contained in  $G$ .*  $\square$

**Lemma 2.6.** (cf. [20]) *In a weak BCC-algebra  $G$  for all  $a, b \in I(G)$  we have  $B(a)B(b) = B(ab)$ .*  $\square$

The identity

$$(4) \quad xy \cdot z = xz \cdot y.$$

plays an important role in the theory of BCI-algebras. It is used in the proofs of many basic facts.

**Definition 2.7.** A weak BCC-algebra  $G$  is called *solid* if the above condition is valid for all  $x, y$  belonging to the same branch and arbitrary  $z \in G$ .

A simple example of a solid weak BCC-algebra is a BCI-algebra. Also BCK-algebra is a solid weak BCC-algebra. A solid BCC-algebra is a BCK-algebra. But there are solid weak BCC-algebras which are not BCI-algebras. The smallest such weak BCC-algebra has 5 elements (cf. [11]).

**Example 2.8.** Consider the set  $X = \{0, 1, 2, 3, 4, 5\}$  with the operation defined by the following table:

$\cdot$	0	1	2	3	4	5
0	0	0	4	4	2	2
1	1	0	4	4	2	2
2	2	2	0	0	4	4
3	3	2	1	0	4	4
4	4	4	2	2	0	0
5	5	4	3	3	1	0

Since  $(S; *, 0)$ , where  $S = \{0, 1, 2, 3, 4\}$ , is a BCI-algebra (see [15]), it is not difficult to verify that  $(X; \cdot, 0)$  is a weak BCC-algebra. It is proper because  $(5 \cdot 3) \cdot 2 \neq (5 \cdot 2) \cdot 3$ . Simple calculations show that this weak BCC-algebra is solid.  $\square$

**Proposition 2.9.** (cf. [11]) *In any solid weak BCC-algebra we have*

$$(a) \quad x \cdot xy \leq y,$$

$$(b) \quad x(x \cdot xy) = xy$$

*for all  $x, y$  belonging to the same branch.*  $\square$

**Corollary 2.10.** *In a solid weak BCC-algebra from  $x, y \in B(a)$  it follows  $x \cdot xy, y \cdot yx \in B(a)$ .*  $\square$

**Proposition 2.11.** (cf. [20]) *The map  $\varphi(x) = 0x$  is an endomorphism of each solid weak BCC-algebra.*  $\square$

**Proposition 2.12.** *In each solid weak BCC-algebra for  $xy, xz$  (or  $xy$  and  $zy$ ) belonging to the same branch we have  $xy \cdot xz \leq zy$ .*

*Proof.* Indeed,  $(xy \cdot xz) \cdot zy = (xy \cdot zy) \cdot xz = 0$ .  $\square$

## 3. COMMUTATIVE SOLID WEAK BCC-ALGEBRAS

In any BCK-algebra  $G$  we can define a binary operation  $\wedge$  by putting

$$x \wedge y = y \cdot yx$$

for all  $x, y \in G$ . A BCK-algebra satisfying the identity

$$(5) \quad x \cdot xy = y \cdot yx,$$

i.e.,  $y \wedge x = x \wedge y$ , is called *commutative*. A commutative BCK-algebra is a lower semilattice with respect to the operation  $\wedge$ .

This definition cannot be extended to BCI-algebras, BCC-algebras and weak BCC-algebras since in any weak BCC-algebra satisfying (5) we have  $0 \cdot 0x = x \cdot x0 = 0$ , i.e.,  $\varphi^2(x) = 0$  for every  $x \in G$ . Thus  $\varphi(x) = \varphi^3(x) = 0$ , by Theorem 2.3. Hence in this algebra  $0 \leq x$  for every  $x \in G$ . This means that this algebra is a commutative BCC-algebra. But in any BCC-algebra  $G$  we have  $0 \leq yx$  for all  $x, y \in G$ , which together with (3) implies  $y \cdot yx \leq y$ . Thus a commutative BCC-algebra satisfies the inequality

$$x \cdot xy = y \cdot yx \leq y.$$

Consequently, it satisfies the identity  $(x \cdot xy)y = 0$ , so it is a BCK-algebra. Hence a commutative (weak) BCC-algebra is a commutative BCK-algebra. Analogously, a commutative BCI-algebra is a commutative BCK-algebra.

But there are weak BCC-algebras in which the condition (5) is satisfied only by elements belonging to the same branch.

**Example 3.1.** A weak BCC-algebra defined by the following table

$\cdot$	0	a	b	c	d
0	0	0	0	c	c
a	a	0	0	c	c
b	b	a	0	d	c
c	c	c	c	0	0
d	d	c	c	a	0

has two branches:  $B(0) = \{0, a, b\}$  and  $B(c) = \{c, d\}$ . It is not difficult to verify that in this weak BCC-algebra (5) is satisfied only by elements belonging to the same branch.  $\square$

**Definition 3.2.** A weak BCC-algebra in which (5) is satisfied by elements belonging to the same branch is called *branchwise commutative*.

**Theorem 3.3.** (cf. [11]) *For a solid weak BCC-algebra  $G$  the following conditions are equivalent:*

- (1)  $G$  is branchwise commutative,
- (2)  $xy = x(y \cdot yx)$  for  $x, y$  from the same branch,
- (3)  $x = y \cdot yx$  for  $x \leq y$ ,
- (4)  $x \cdot xy = y(y(x \cdot xy))$  for  $x, y$  from the same branch,
- (5) each branch of  $G$  is a semilattice with respect to the operation  $x \wedge y = y \cdot yx$ .

$\square$

In the proof of the next theorem we will need the following well-known result from the theory of BCK-algebras.

**Lemma 3.4.** *If  $p$  is the greatest element of a commutative BCK-algebra  $G$ , then  $(G; \leq)$  is a distributive lattice with respect to the operations  $x \wedge y = y \cdot yx$  and  $x \vee y = p(px \wedge py)$ .  $\square$*

**Theorem 3.5.** *In a solid branchwise commutative weak BCC-algebra  $G$ , for every  $p \in G$ , the set  $A(p) = \{x \in G : x \leq p\}$  is a distributive lattice with respect to the operations  $x \wedge y = y \cdot yx$  and  $x \vee_p y = p(px \wedge py)$ .*

*Proof.* (A). We prove that  $(A^p; \leq)$ , where

$$A^p = \{px : x \in A(p)\},$$

is a distributive lattice.

First, we show that  $A^p$  is a subalgebra of  $B(0)$ . It is clear that  $0 = pp \in A^p$  and  $A(p) \subseteq B(a)$  for some  $a \in I(G)$ . Thus  $A^p \subset B(0)$ . Obviously  $a \leq x$  for every  $x \in A(p)$ . Consequently,  $px \leq pa$ . Hence  $pa$  is the greatest element of  $A^p$ .

Let  $px, py$  be arbitrary elements of  $A^p$ . Then obviously  $yx \in B(0)$  and  $z = p \cdot yx \in A^p$  because  $zp = (p \cdot yx)p = 0$ . Moreover,  $zx = (p \cdot yx)x = px \cdot yx \leq py$ , by (i'). Since

$$(px \cdot pz) \cdot zx = (px \cdot zx) \cdot pz = 0,$$

we also have  $px \cdot pz \leq zx \leq py$ . Therefore,  $0 = (px \cdot pz) \cdot py = (px \cdot py) \cdot pz$ , i.e.,

$$(6) \quad px \cdot py \leq pz.$$

On the other hand, since a weak BCC-algebra  $G$  is branchwise commutative, for every  $y \in A(p)$ , according to Theorem 3.3, we have  $p \cdot py = y$ . Hence  $px \cdot py = (p \cdot py)x = yx$ . But  $pz \cdot yx = p(p \cdot yx) \cdot yx = (p \cdot yx)(p \cdot yx) = 0$ . Thus  $pz \leq yx = px \cdot py$ , which together with (6) gives

$$px \cdot py = pz.$$

Hence  $A^p$  is a subalgebra of  $B(0)$ . Obviously,  $B(0)$  as a BCC-algebra contained in  $G$  is commutative, and consequently it is a commutative BCK-algebra. Thus  $A^p$  is a commutative BCK-algebra, too. By Lemma 3.4,  $(A^p; \leq)$  is a distributive lattice.

(B). Now we show that  $(A(p); \leq)$  is a distributive lattice. Clearly,  $p$  is the greatest element of  $A(p)$ .

Let  $x, y \in A(p)$ . Then  $px, py \in A^p$  and from the fact that  $(A^p; \leq)$  is a lattice it follows that there exists the last upper bound  $pz \in A^p$ , i.e.,

$$(7) \quad px \vee_p py = pz.$$

Observe that for  $x, y \in A(p)$  we have

$$(8) \quad py \leq px \iff x \leq y.$$

Indeed, in view of (3),  $x \leq y$  implies  $py \leq px$ . Similarly,  $py \leq px$  implies  $p \cdot px \leq p \cdot py$ . But  $G$  is branchwise commutative, hence by Theorem 3.3, for every  $v \in A(p)$  we have  $p \cdot pv = v$ . Therefore  $x = p \cdot px \leq p \cdot py = y$ .

From (7) and (8) it follows that  $z$  is the greatest lower bound for  $x$  and  $y$ . Hence  $x \wedge y = z$ . Moreover, we have  $p(x \wedge y) = pz$  and  $px \vee_p py = pz$ , which implies

$$(9) \quad p(x \wedge y) = px \vee_p py.$$

Analogously, we can prove that for all  $x, y \in A(p)$  there exists  $x \vee_p y$  and

$$(10) \quad p(x \vee_p y) = px \wedge py.$$

Therefore  $(A(p); \leq)$  is a lattice.

Since (9) and (10) are satisfied in  $A^p$  and  $(A^p; \leq)$  is a distributive lattice, we have

$$px \vee_p (py \wedge pz) = (px \vee_p py) \wedge (px \vee_p pz)$$

for all  $px, py, pz \in A^p$ .

This, in view of (9) and (10), gives

$$p(x \wedge (y \vee_p z)) = p((x \wedge y) \vee_p (x \wedge z)).$$

Consequently,

$$p \cdot p(x \wedge (y \vee_p z)) = p \cdot p((x \wedge y) \vee_p (x \wedge z))$$

and

$$x \wedge (y \vee_p z) = (x \wedge y) \vee_p (x \wedge z),$$

because  $x \wedge (y \vee_p z), (x \wedge y) \vee_p (x \wedge z) \in A(p)$ . This means that  $(A(p); \leq)$  is a distributive lattice.

In this lattice  $x \vee_p y = p(p(x \vee_p y)) = p(px \vee_p py)$ .

This completes the proof.  $\square$

**Definition 3.6.** A weak BCC-algebra  $G$  is called *restricted*, if every its branch has the greatest element.

The greatest element of the branch  $B(a)$  will be denoted by  $1_a$ . By  $N_a$  we will denote the unary operation  $N_a : G \rightarrow G$  defined by  $N_ax = 1_ax$ .

**Lemma 3.7.** *The main properties of the operation  $N_a$  in restricted solid weak BCC-algebras are as follows:*

- (1)  $N_a 1_a = 0$  and  $N_a 0 = 1_a$ ,
- (2)  $N_a N_a x \leq x$ ,
- (3)  $(N_a x)y = (N_a y)x$ ,
- (4)  $x \leq y \implies N_a y \leq N_a x$ ,
- (5)  $N_a x N_a y \leq yx$ ,
- (6)  $N_a N_a N_a x = N_a x$ ,

where  $x, y \in B(a)$ .  $\square$

**Definition 3.8.** A restricted weak BCC-algebra  $G$  is called *involutory*, if  $N_a N_a x = x$  holds for every  $x \in B(a)$ . An element  $x$  satisfying this condition is called an *involution*.

A simple example of involutions in restricted solid weak BCC-algebras are  $a \in I(G)$  and  $1_a$ .

In an involutory weak BCC-algebra the map  $N_a : B(a) \rightarrow B(0)$  is one-to-one. Thus in an involutory weak BCC-algebra with finite  $B(0)$  all branches are finite.

**Proposition 3.9.** *Any branchwise commutative restricted solid weak BCC-algebra is involutory.*

*Proof.* Indeed,  $N_a N_a x = 1_a(1_a x) = x(1_a x) = x0 = x$  for every  $x \in B(a)$ .  $\square$

**Lemma 3.10.** *In an involutory solid weak BCC-algebra*

$$xy = N_a y N_a x$$

*is valid for all  $a \in I(G)$  and  $x, y \in B(a)$ .*

*Proof.* In fact,  $xy = (N_a N_a x)y = (1_a \cdot 1_a x)y = 1_a y \cdot 1_a x = N_a y N_a x$ .  $\square$

**Proposition 3.11.** *A solid weak BCC-algebra is involutory if and only if*

$$xN_ay = yN_ax$$

*holds for all  $a \in I(G)$  and  $x, y \in B(a)$ .*

*Proof.* Clearly  $N_ax, N_ay \in B(0)$  for  $x, y \in B(a)$ . Thus  $yN_ax, xN_ay \in B(a)$ , and consequently

$$xN_ay \cdot yN_ax = (x \cdot 1_ay)(y \cdot 1_ax) = x(y \cdot 1_ax) \cdot 1_ay = (1_a \cdot 1_ax)(y \cdot 1_ax) \cdot 1_ay.$$

Since  $(1_a \cdot 1_ax)(y \cdot 1_ax) \leq 1_ay$ , by (i'), from (2) it follows

$$(1_a \cdot 1_ax)(y \cdot 1_ax) \cdot 1_ay \leq 1_ay \cdot 1_ay = 0.$$

Hence  $xN_ay \cdot yN_ax = 0$ . Analogously we show  $yN_ax \cdot xN_ay = 0$ , which by (iv) implies  $xN_ay = yN_ax$ .

On the other hand, if  $xN_ay = yN_ax$  holds for all  $a \in I(G)$  and  $x, y \in B(a)$ , then for  $y = 1_a$  and arbitrary  $x \in B(a)$  we have

$$x = x \cdot 1_a 1_a = xN_a 1_a = 1_a N_ax = N_a N_ax,$$

which means that this weak BCC-algebra is involutory.  $\square$

**Theorem 3.12.** *For an involutory solid weak BCC-algebra  $G$  the following statements are equivalent:*

- (1) *each branch of  $G$  is a lower semilattice,*
- (2) *each branch of  $G$  is a lattice.*

*Moreover, if  $(B(a); \leq)$  is a lattice, then*

$$x \wedge y = N_a(N_ax \vee_a N_ay) \quad \text{and} \quad x \vee_a y = N_a(N_ax \wedge N_ay).$$

*Proof.* (1)  $\implies$  (2) Since each branch  $B(a)$  of  $G$  is a lower semilattice, then  $N_ax \wedge N_ay$  exists for all  $x, y \in B(a)$ , i.e., for all  $N_ax, N_ay \in B(0)$ . Hence  $N_ax \wedge N_ay \leq N_ax$ , which gives  $z(N_ax \wedge N_ay) \leq zN_ax$  for each  $z \in B(a)$ . Similarly,  $zN_ay \leq z(N_ax \wedge N_ay)$ . This means that  $z(N_ax \wedge N_ay)$  is an upper bound of  $zN_ax$  and  $zN_ay$ . Let us assume that  $u \in B(a)$  is another upper bound for  $zN_ax$  and  $zN_ay$ . From  $zN_ax \leq u$  and  $zN_ay \leq u$  we have  $zu \leq z(zN_ax)$  and  $zu \leq z(zN_ay)$ . But  $z(zN_ax) \leq N_ax$  for  $v, z \in B(a)$ , because  $N_av \in B(0)$  and  $zN_av \in B(a)$ . Hence  $zu \leq N_ax$  and  $zu \leq N_ay$ , which implies  $zu \leq N_ax \wedge N_ay$ . Thus  $z(N_ax \wedge N_ay) \leq z(zu) \leq u$  and  $z(N_ax \wedge N_ay)$  is the least upper bound of  $zN_ax$  and  $zN_ay$ . Therefore for every  $x, y, z \in B(a)$  there exists the least upper bound of  $zN_ax$  and  $zN_ay$ , i.e.,  $zN_ax \vee_a zN_ay$ . In particular, for every  $x, y \in B(a)$  there exists

$$1_a N_ax \vee_a 1_a N_ay = N_a N_ax \vee_a N_a N_ay = x \vee_a y.$$

This shows that  $(B(a); \leq)$  is an upper semilattice. Consequently,  $B(a)$  is a lattice.

(2)  $\implies$  (1) Obvious.

Since  $(B(a); \leq)$  is a lattice for every  $a \in I(G)$ , using the same argumentation as in the second part of the proof of Theorem 3.5 we can show that  $N_a(x \wedge y) = N_ax \vee_a N_ay$  for  $x, y \in B(a)$ . Thus,

$$x \wedge y = N_a N_a(x \wedge y) = N_a(N_ax \vee_a N_ay).$$

Analogously,  $N_a(x \vee_a y) = N_ax \wedge N_ay$  implies

$$x \vee_a y = N_a N_a(x \vee_a y) = N_a(N_ax \wedge N_ay).$$

This completes the proof.  $\square$



## 4. N-FOLD BRANCHWISE COMMUTATIVE WEAK BCC-ALGEBRAS

In a weak BCC-algebra  $G$  for all  $x, y \in G$  we put  $xy^0 = x$  and  $xy^{n+1} = (xy^n)y$  for any non-negative integer  $n$ .

**Definition 4.1.** A weak BCC-algebra  $G$  is called *n-fold branchwise commutative* (shortly: *n-b commutative*), if there exists a natural number  $n$  such that

$$(11) \quad xy = x(y \cdot yx^n)$$

holds for  $x, y$  belonging to the same branch.

From Theorem 3.3 it follows that for  $n = 1$  it is an ordinary branchwise commutative weak BCC-algebra.

**Theorem 4.2.** For a solid weak BCC-algebra  $G$  the following conditions are equivalent:

- (a)  $G$  is *n-b commutative*,
- (b)  $x \cdot xy \leq y \cdot yx^n$  for  $x, y$  belonging to the same branch,
- (c)  $x \leq y \implies x \leq y \cdot yx^n$ .

*Proof.* (a)  $\implies$  (b) Let  $x, y \in B(a)$  for some  $a \in B(a)$ . Then  $xy \in B(0)$  and consequently  $x(y \cdot yx^n) \in B(0)$ . So,  $x$  and  $y \cdot yx^n$  are in the same branch. Hence  $y \cdot yx^n \in B(a)$  and

$$(x \cdot xy)(y \cdot yx^n) = x(y \cdot yx^n) \cdot xy = xy \cdot xy = 0.$$

Therefore  $x \cdot xy \leq y \cdot yx^n$ .

(b)  $\implies$  (c) Obvious.

(c)  $\implies$  (a) Since  $0 \leq xy$  for  $x, y \in B(a)$ , we have

$$(12) \quad x \cdot xy \leq x.$$

Consequently,  $yx \leq y(x \cdot xy)$ . This implies

$$yx \cdot x \leq y(x \cdot xy) \cdot x = yx \cdot (x \cdot xy)$$

and

$$yx \cdot (x \cdot xy) \leq y(x \cdot xy) \cdot (x \cdot xy).$$

Therefore

$$yx^2 = yx \cdot x \leq y(x \cdot xy) \cdot (x \cdot xy) = y(x \cdot xy)^2,$$

i.e.,

$$yx^2 \leq y(x \cdot xy)^2.$$

From this inequality we obtain

$$(13) \quad yx^2 \cdot x \leq y(x \cdot xy)^2 \cdot x.$$

From (12) we get

$$y(x \cdot xy)^2 \cdot x \leq y(x \cdot xy)^2 \cdot (x \cdot xy),$$

which together with (13) gives

$$yx^3 \leq y(x \cdot xy)^3.$$

Repeating the above procedure we can see that

$$yx^n \leq y(x \cdot xy)^n$$

holds for every natural  $n$ . Hence

$$(14) \quad x(y \cdot yx^n) \leq x(y \cdot y(x \cdot xy)^n).$$

Obviously  $x \cdot xy \leq y$  for  $x, y$  belonging to the same branch. Applying (c) to the last inequality we obtain  $x \cdot xy \leq y \cdot y(x \cdot xy)^n$ . Hence, by (3) and Proposition 2.9 (b), we conclude  $x(y \cdot y(x \cdot xy)^n) \leq x(x \cdot xy) = xy$ . Consequently,

$$(15) \quad x(y \cdot y(x \cdot xy)^n) \leq xy.$$

Combining (14) and (15) we get

$$(16) \quad x(y \cdot yx^n) \leq xy.$$

Thus  $x(y \cdot yx^n) \in B(0)$ . This means that  $y \cdot yx^n \in B(a)$ . But in this case  $yx^n \in B(0)$ . Indeed, if  $yx^n \in B(b)$  for some  $b \in I(G)$ , then  $y \cdot yx^n \in B(a)B(b) = B(ab)$ . So,  $y \cdot yx^n \in B(a) \cap B(ab)$ . Thus  $B(a) = B(ab)$ , i.e.,  $a = ab$ . Hence  $0 = ab \cdot a = aa \cdot b = 0b$ . Therefore  $b \in B(0)$  and  $b = 0$  because  $b \in I(G)$ .

This means that

$$(y \cdot yx^n)y = 0 \cdot yx^n = 0.$$

Consequently,  $y \cdot yx^n \leq y$ , which, by (3), implies  $xy \leq x(y \cdot yx^n)$ .

Comparing the last inequality with (16) we obtain  $xy = x(y \cdot yx^n)$ . This completes the proof.  $\square$

## 5. IMPLICATIVE SOLID WEAK BCC-ALGEBRAS

Implicative and positive implicative BCC-algebras are originating from the systems of positive implicational calculus and weak positive implicational calculus in the implicational functor in logical systems. In this section we will also deal with some generalized implicative and positive implicative solid weak BCC-algebras.

**Definition 5.1.** A weak BCC-algebra  $G$  is called *branchwise implicative*, if

$$x \cdot yx = x$$

holds for all  $x, y$  belonging to the same branch of  $G$ .

**Theorem 5.2.** ([11], Theorem 3.8) *Any solid branchwise implicative weak BCC-algebra is branchwise commutative.*  $\square$

**Theorem 5.3.** *In a solid branchwise implicative weak BCC-algebra the equation*

$$xy \cdot 0y = (xy \cdot 0y)y \cdot 0y$$

*is satisfied by all elements belonging to the same branch.*

*Proof.* Let  $G$  be branchwise implicative and solid. Then, according to (i), for all  $x, y \in B(a)$ , we have

$$(17) \quad (xy \cdot 0y)x = 0.$$

So,  $xy \cdot 0y \in B(a)$  and  $xy, x(xy \cdot 0y) \in B(0)$ . Therefore,

$$\begin{aligned} (xy \cdot 0y) \cdot x(xy \cdot 0y) &= (xy \cdot x(xy \cdot 0y)) \cdot 0y \\ &= (x \cdot x(xy \cdot 0y))y \cdot 0y \\ &= ((xy \cdot 0y) \cdot (xy \cdot 0y)x)y \cdot 0y && \text{by Theorem 5.2} \\ &= ((xy \cdot 0y)0)y \cdot 0y && \text{by (17)} \\ &= (xy \cdot 0y)y \cdot 0y. \end{aligned}$$

Hence

$$(xy \cdot 0y) \cdot x(xy \cdot 0y) = (xy \cdot 0y)y \cdot 0y.$$

Since  $xy \cdot 0y$  and  $x$  are in the same branch, the implicativity shows that

$$(xy \cdot 0y) \cdot x(xy \cdot 0y) = xy \cdot 0y,$$

which together with the previous equation gives  $xy \cdot 0y = (xy \cdot 0y)y \cdot 0y$ .  $\square$

**Lemma 5.4.** *In a solid weak BCC-algebra for  $x, y$  belonging to the same branch holds*

$$(xy \cdot 0y)y \cdot 0y \leq ((xy \cdot y) \cdot 0y) \cdot 0y.$$

*Proof.* Indeed, if  $x, y \in B(a)$ , then as in the previous proof we can also see that  $xy \cdot 0y \in B(a)$ . Hence

$$\begin{aligned} ((xy \cdot 0y)y \cdot 0y) \cdot (((xy \cdot y) \cdot 0y) \cdot 0y) &\leq (xy \cdot 0y)y \cdot ((xy \cdot y) \cdot 0y) && \text{by (i')} \\ &= ((xy \cdot 0y)((xy \cdot y) \cdot 0y)) \cdot y \\ &\leq (xy \cdot (xy \cdot y)) \cdot y && \text{by (i')} \\ &\leq (x \cdot xy) \cdot y = xy \cdot xy = 0, \end{aligned}$$

i.e.,

$$((xy \cdot 0y)y \cdot 0y) \cdot (((xy \cdot y) \cdot 0y) \cdot 0y) = 0.$$

This implies

$$((xy \cdot 0y)y) \cdot 0y \leq ((xy \cdot y) \cdot 0y) \cdot 0y.$$

The proof is complete.  $\square$

**Theorem 5.5.** *If  $I(G)$  is a BCK-ideal of a branchwise commutative solid weak BCC-algebra  $G$  and*

$$(18) \quad xy \cdot 0y = ((xy \cdot y) \cdot 0y) \cdot 0y$$

*is valid for all  $x, y$  belonging to the same branch, then  $G$  is branchwise implicative.*

*Proof.* Let  $x, y \in B(a)$  for some  $a \in I(G)$ . Then

$$\begin{aligned} x(x \cdot yx) \cdot 0x &= (x(x \cdot yx) \cdot 0) \cdot 0x \\ &= (x(x \cdot yx) \cdot (x \cdot yx)(x \cdot yx)) \cdot 0x \\ &= (x(x \cdot yx) \cdot (x(x \cdot yx) \cdot yx)) \cdot 0x && \text{since } x, x \cdot yx \in B(a) \\ &= (yx \cdot (yx \cdot x(x \cdot yx))) \cdot 0x, \end{aligned}$$

because  $yx, x(x \cdot yx) \in B(0)$  and  $G$  is branchwise commutative. But  $B(0)$  is a subalgebra of  $G$  (Lemma 2.5), hence  $yx \cdot x(x \cdot yx) \in B(0)$ . Therefore

$$\begin{aligned} (yx \cdot (yx \cdot x(x \cdot yx))) \cdot 0x &= (yx \cdot 0x) \cdot (yx \cdot x(x \cdot yx)) \\ &= (((yx \cdot x) \cdot 0x) \cdot 0x) \cdot (yx \cdot x(x \cdot yx)), \end{aligned}$$

by (18).

Since  $(yx \cdot x) \cdot 0x \leq yx \cdot 0 = yx \in B(0)$  implies  $(yx \cdot x) \cdot 0x \in B(0)$ , from the above we obtain

$$\begin{aligned} x(x \cdot yx) \cdot 0x &= (((yx \cdot x) \cdot 0x) \cdot 0x) \cdot (yx \cdot x(x \cdot yx)) \\ &= (((yx \cdot x) \cdot 0x) \cdot (yx \cdot x(x \cdot yx))) \cdot 0x \\ &= (((yx \cdot x) \cdot (yx \cdot x(x \cdot yx))) \cdot 0x) \cdot 0x, \end{aligned}$$

because, as it is not difficult to see,  $yx \cdot x, 0x \in B(0a)$ .

Now from the fact that  $yx \cdot x(x \cdot yx)$  and  $x(x \cdot yx)$  are in  $B(0)$  and  $G$  is branchwise commutative we have

$$\begin{aligned} x(x \cdot yx) \cdot 0x &= (((yx \cdot x) \cdot (yx \cdot x(x \cdot yx))) \cdot 0x) \cdot 0x \\ &= (((yx \cdot (yx \cdot x(x \cdot yx))) \cdot x) \cdot 0x) \cdot 0x \\ &= (((x(x \cdot yx) \cdot (x(x \cdot yx) \cdot yx)) \cdot x) \cdot 0x) \cdot 0x. \end{aligned}$$

From this, in view of  $x \cdot yx \in B(a)$ , we get

$$\begin{aligned} x(x \cdot yx) \cdot 0x &= (((x(x \cdot yx) \cdot ((x \cdot xy)(x \cdot yx))) \cdot x) \cdot 0x) \cdot 0x \\ &= (((x(x \cdot yx) \cdot 0) \cdot x) \cdot 0x) \cdot 0x \\ &= ((x(x \cdot yx))x \cdot 0x) \cdot 0x \\ &= ((xx \cdot (x \cdot yx)) \cdot 0x) \cdot 0x \\ &= ((0 \cdot (x \cdot yx)) \cdot 0x) \cdot 0x \\ &= ((0x \cdot (0 \cdot yx)) \cdot 0x) \cdot 0x && \text{by Proposition 2.11} \\ &= ((0x \cdot 0) \cdot 0x) \cdot 0x \\ &= (0x \cdot 0x) \cdot 0x = 0 \cdot 0x \in I(G), \end{aligned}$$

because  $I(G) = \varphi(G)$ . Hence  $x(x \cdot yx) \cdot 0x \in I(G)$ . Also  $0x \in I(G)$ . Since, by the assumption,  $I(G)$  is a BCK-ideal of  $G$ , we obtain  $x(x \cdot yx) \in I(G)$ . But  $x(x \cdot yx) \in B(0)$ , so  $x(x \cdot yx) \in I(G) \cap B(0)$ . Thus  $x(x \cdot yx) = 0$ . This means that  $x \leq x \cdot yx \leq x$ . Consequently,  $x \cdot yx = x$ . Therefore  $G$  is branchwise implicative. The proof is complete.  $\square$

The example presented below shows that in the last theorem the assumption on  $I(G)$  is essential.

**Example 5.6.** Consider a weak BCC-algebra  $G$  defined by the following table:

$\cdot$	0	1	2	3	4
0	0	0	0	3	3
1	1	0	0	3	3
2	2	1	0	4	4
3	3	3	3	0	0
4	4	3	3	1	0

Because  $(S; \cdot, 0)$ , where  $S = \{0, 1, 3, 4\}$ , is a BCI-algebra (see [15], p.337) to show that  $G$  is a weak BCC is sufficient to check the axiom (i) in the case when at least one of the elements  $x, y, z$  is equal to 2. Such defined weak BCC-algebra is proper

since  $23 \cdot 4 \neq 24 \cdot 3$ . It also is branchwise commutative and satisfies (18) but it is not branchwise implicative. Obviously  $I(G)$  is not a BCK-ideal of  $G$ .  $\square$

## 6. POSITIVE IMPLICATIVE WEAK BCC-ALGEBRAS

As it is well-known a BCK-algebra is called *positive implicative*, if it satisfies the identity

$$(19) \quad xy \cdot y = xy.$$

In BCK-algebras this identity is equivalent to

$$(20) \quad xy \cdot z = xz \cdot yz.$$

Positive implicative BCC-algebras can be defined in the same way (cf. [8] or [9]), however weak BCC-algebras cannot because by putting  $x = y$  in (19) we obtain  $0x = 0$  for every  $x \in G$ . This means that a weak BCC-algebra (as well as BCI-algebra) satisfying (19) or (20) is a BCC-algebra. Therefore positive implicative weak BCC-algebras and BCI-algebras should be defined in another way. One way was proposed by J. Meng and X. L. Xin in [22]. They defined a positive implicative BCI-algebra as a BCI-algebra satisfying the identity  $xy = (xy \cdot y) \cdot 0y$ . (Equivalent conditions one can find in [22] and [15].) Using this definition it can be proved that a BCI-algebra is implicative if and only if it is both positive implicative and commutative. Unfortunately, in the proof of this result a very important role plays the identity (4). So, this proof can not be transferred to weak BCC-algebras. In connection with this fact, W.A. Dudek introduced in [11] the new class of positive implicative weak BCC-algebras called by him  $\varphi$ -implicative.

**Definition 6.1.** A weak BCC-algebra  $G$  is called  $\varphi$ -implicative, if it satisfies the identity

$$(21) \quad xy = xy \cdot y(0 \cdot 0y),$$

i.e.,

$$xy = xy \cdot y\varphi^2(y).$$

If (21) is satisfied only by elements belonging to the same branch, then we say that this weak BCC-algebra is *branchwise  $\varphi$ -implicative*.

It is clear that in the case of BCC-algebras the conditions (21) and (19) are equivalent. Thus a BCC-algebra is  $\varphi$ -implicative if and only if it is positive implicative. For BCI-algebras and weak BCC-algebras it is not true. A group-like weak BCC-algebra determined by a group, i.e., a weak BCC-algebra  $(G; \cdot, 0)$  with the operation  $xy = x * y^{-1}$  where  $(G; *, 0)$  is a group, is a simple example of a  $\varphi$ -implicative weak BCC-algebra which is not positive implicative.

**Definition 6.2.** A weak BCC-algebra  $G$  is called *weakly positive implicative*, if it satisfies the identity

$$(22) \quad xy \cdot z = (xz \cdot z) \cdot yz.$$

If (22) is satisfied only by elements belonging to the same branch, then we say that this weak BCC-algebra is *branchwise weakly positive implicative*.

**Example 6.3.** Routine and easy calculations show that a weak BCC-algebra defined by the following table:

$\cdot$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

is weakly positive implicative.  $\square$

**Lemma 6.4.** *A solid weakly positive implicative weak BCC-algebra  $G$  satisfies the identity*

$$(23) \quad xy = (xy \cdot y) \cdot 0y.$$

*Proof.* By putting  $y = 0$  in (22) we obtain the identity  $xz = (xz \cdot z) \cdot 0z$ , which is equivalent to (23).  $\square$

**Theorem 6.5.** *A solid weakly positive implicative weak BCC-algebra is branchwise  $\varphi$ -implicative.*

*Proof.* Let  $G$  be a solid weakly positive implicative weak BCC-algebra. Then for  $x, y \in B(a)$ ,  $a \in I(G)$  and  $\varphi(x) = 0x$  we have

$$\begin{aligned} (xy \cdot y(0 \cdot 0y)) \cdot xy &= (xy \cdot y\varphi^2(y)) \cdot xy = (xy \cdot xy) \cdot y\varphi^2(y) \\ &= 0 \cdot y\varphi^2(y) = \varphi(y)\varphi^3(y) = \varphi(y)\varphi(y) = 0. \end{aligned}$$

Hence

$$(24) \quad xy \cdot y(0 \cdot 0y) \leq xy.$$

On the other hand,

$$\begin{aligned} xy \cdot (xy \cdot y(0 \cdot 0y)) &= xy \cdot (xy \cdot y\varphi^2(y)) \\ &= ((xy \cdot y) \cdot 0y) \cdot (xy \cdot y\varphi^2(y)) && \text{by (23)} \\ &= (xy \cdot y)\varphi(y) \cdot (xy \cdot y\varphi^2(y)) \\ &= (xy \cdot y)(xy \cdot y\varphi^2(y)) \cdot \varphi(y) \\ &= (xy \cdot (xy \cdot y\varphi^2(y)))y \cdot \varphi(y), \end{aligned}$$

because, according to Lemma 2.6, we have  $xy \cdot y, \varphi(y) \in B(0a)$  and  $xy, xy \cdot y\varphi^2(y) \in B(0)$ . Since in this case also  $y\varphi^2(y) \in B(0)$ , therefore

$$(xy \cdot (xy \cdot y\varphi^2(y))) \cdot y\varphi^2(y) = (xy \cdot y\varphi^2(y)) \cdot (xy \cdot y\varphi^2(y)) = 0.$$

Thus

$$(xy \cdot (xy \cdot y\varphi^2(y))) \leq y\varphi^2(y),$$

which, by (2), implies

$$((xy \cdot (xy \cdot y\varphi^2(y))))y \cdot \varphi(y) \leq (y\varphi^2(y))y \cdot \varphi(y).$$

Hence

$$\begin{aligned}
xy \cdot (xy \cdot y(0 \cdot 0y)) &= (xy \cdot (xy \cdot y\varphi^2(y)))y \cdot \varphi(y) \\
&\leq (y\varphi^2(y))y \cdot \varphi(y) \\
&= (yy \cdot \varphi^2(y)) \cdot \varphi(y) \\
&= (0 \cdot \varphi^2(y)) \cdot \varphi(y) \\
&= \varphi^3(y) \cdot \varphi(y) = 0
\end{aligned}$$

because  $\varphi^3(y) = \varphi(y)$  by Theorem 2.3.

This proves

$$(25) \quad xy \leq xy \cdot y(0 \cdot 0y)$$

Combining (24) and (25) we get

$$xy = xy \cdot y(0 \cdot 0y).$$

So,  $G$  is a solid branchwise  $\varphi$ -implicative weak BCC-algebra.  $\square$

The converse of the Theorem 6.5 is not true.

**Example 6.6.** It is not difficult to see that the following weak BCC-algebra is proper and solid.

$\cdot$	0	1	2	3	4
0	0	0	0	3	3
1	1	0	0	3	3
2	2	2	0	4	4
3	3	3	3	0	0
4	4	4	3	1	0

It is branchwise  $\varphi$ -implicative but not weakly positive implicative since  $4 \cdot 3 = 1$  and  $((4 \cdot 3) \cdot 3) \cdot (0 \cdot 3) = 0$ .  $\square$

**Theorem 6.7.** *A solid weak BCC-algebra is branchwise implicative if and only if it is branchwise  $\varphi$ -implicative and branchwise commutative.*

*Proof.* Let  $G$  be a branchwise implicative solid weak BCC-algebra. Then  $x = x \cdot yx$  for  $x, y \in B(a)$ . Consequently,

$$(26) \quad x \cdot xy = (x \cdot yx) \cdot xy.$$

Since  $x \cdot yx, y \cdot yx \in B(a)$ , we have

$$(x \cdot xy) \cdot (y \cdot yx) \stackrel{(26)}{=} ((x \cdot yx) \cdot xy) \cdot (y \cdot yx) \stackrel{(4)}{=} (x \cdot yx)(y \cdot yx) \cdot xy = 0,$$

by (i). Hence  $x \cdot xy \leq y \cdot yx$ . Thus  $x \cdot xy = y \cdot yx$ , which shows that  $G$  is branchwise commutative.

Next, we obtain

$$\begin{aligned}
(xy \cdot y\varphi^2(y)) \cdot xy &= (xy \cdot xy) \cdot y\varphi^2(y) = 0 \cdot y\varphi^2(y) \\
&= 0y \cdot 0\varphi^2(y) = \varphi(y) \cdot \varphi^3(y) = \varphi(y) \cdot \varphi(y) = 0,
\end{aligned}$$

because  $\varphi$  is an endomorphism (Proposition 2.11) such that  $\varphi^3 = \varphi$  (Theorem 2.3). Thus

$$(27) \quad xy \cdot y\varphi^2(y) \leq xy.$$

Moreover, from the the fact that a weak BCC-algebra  $G$  is branchwise commutative and  $xy, y\varphi^2(y) \in B(0)$ , we obtain

$$\begin{aligned} xy \cdot (xy \cdot y\varphi^2(y)) &= y\varphi^2(y) \cdot (y\varphi^2(y) \cdot xy) \\ &= y\varphi^2(y) \cdot (y \cdot xy)\varphi^2(y) && \text{since } \varphi^2(y) \in B(a) \\ &= y\varphi^2(y) \cdot y\varphi^2(y) && \text{since } y \cdot xy = y \\ &= 0. \end{aligned}$$

Hence

$$(28) \quad xy \leq xy \cdot y\varphi^2(y).$$

Comparing (27) and (28) we get  $xy = yx \cdot y\varphi^2(y)$ , so this weak BCC-algebra is  $\varphi$ -implicative.

Conversely, let a solid weak BCC-algebra  $G$  be branchwise  $\varphi$ -implicative and branchwise commutative. Then  $x \cdot yx \in B(a)$  for any  $x, y \in B(a)$ . Hence

$$x(x \cdot yx) \cdot yx = (x \cdot yx)(x \cdot yx) = 0.$$

Consequently,

$$x(x \cdot yx) = x(x \cdot yx) \cdot 0 = x(x \cdot yx) \cdot (x(x \cdot yx) \cdot yx).$$

But  $yx$  and  $x(x \cdot yx)$  are in  $B(0)$  and  $G$  is branchwise commutative, so we also have

$$x(x \cdot yx) \cdot (x(x \cdot yx) \cdot yx) = yx \cdot (yx \cdot x(x \cdot yx)).$$

Thus

$$x(x \cdot yx) = yx \cdot (yx \cdot x(x \cdot yx)).$$

Since, by Lemma 2.6, elements  $yx, yx \cdot x\varphi^2(x)$  and  $yx \cdot x(x \cdot yx)$  are in  $B(0)$ , from the above, in view of  $\varphi$ -implicativity of  $G$  and Proposition 2.12, we obtain

$$\begin{aligned} x(x \cdot yx) &= (yx \cdot x\varphi^2(x)) \cdot (yx \cdot x(x \cdot yx)) \\ &\leq x(x \cdot yx) \cdot x\varphi^2(x) \leq \varphi^2(x)(x \cdot yx), \end{aligned}$$

because  $x(x \cdot yx), x\varphi^2(x) \in B(0)$ .

Moreover, from  $\varphi^2(x) \in B(a)$  we get  $a \leq \varphi^2(x)$ , which, by Theorem 2.3, implies  $a = \varphi^2(a) = \varphi^4(x) = \varphi^2(x)$ . Thus

$$x(x \cdot yx) \leq \varphi^2(x)(x \cdot yx) = a(x \cdot yx) = 0,$$

because  $x \cdot yx \in B(a)$ . Hence  $x \leq x \cdot yx$ .

On the other hand,  $(x \cdot yx)x = xx \cdot yx = 0 \cdot yx = 0$ , which together with the previous inequality gives  $x \cdot yx = x$ .

This completes the proof.  $\square$

## 7. WEAK BCC-ALGEBRAS WITH CONDITION (S)

BCK-algebras with condition (S) were introduced by K. Iséki in [18] and next generalized to BCI-algebras. Later such algebras were extensively studied by several authors from different points of view. Today BCK-algebras with condition (S) are an important class of BCK-algebras.

Below we extend this concept to the case of weak BCC-algebras and prove basic properties of these algebras.



For given two elements  $x$  and  $y$  of a weak BCC-algebra  $G$  we consider the set

$$A(x, y) = \{p \in G : px \leq y\} = \{p \in G : px \cdot y = 0\}.$$

We start with the following simple lemma.

**Lemma 7.1.** *Let  $G$  be a weak BCC-algebra. Then for  $x, y, z, u \in G$  we have*

- (1)  $A(0, x) = A(x, 0)$ ,
- (2)  $0 \in A(x, y) \iff 0 \in A(y, x)$ ,
- (3)  $x \in A(x, y) \iff y \in B(0)$ ,
- (4)  $x \in B(0) \implies y \in A(x, y)$ ,
- (5)  $A(x, y) \subset A(u, y)$  for  $x \leq u$ ,
- (6)  $A(x, y) \subset A(x, z)$  for  $y \leq z$ ,
- (7)  $u \leq z, z \in A(x, y) \implies u \in A(x, y)$ ,
- (8)  $A(x, y) = A(y, x)$  if  $G$  is a BCI-algebra. □

Example 6.6 shows that in general  $A(x, y) \neq A(y, x)$ . Indeed, in a weak BCC-algebra defined in this example  $A(3, 4) \neq A(4, 3)$ , 3, 4 are not in  $A(3, 4)$  and  $3 \in A(2, 3)$  does not imply  $2 \in B(0)$ .

**Proposition 7.2.** *Let  $G$  be a solid weak BCC-algebra. If  $x \in B(a)$ ,  $y \in B(b)$ , then  $A(x, y)$  is a non-empty subset of the branch  $B(a \cdot 0b)$ .*

*Proof.* Let  $x \in B(a)$ ,  $y \in B(b)$ . Then  $\varphi^2(x) \in I(G) \cap B(a)$  and  $\varphi(y) = \varphi(b)$ , by Theorem 2.3. Hence  $\varphi^2(x) = a$ . Moreover, since  $\varphi$  is an endomorphism (Proposition 2.11), we have

$$s = 0(0x \cdot y) = \varphi(\varphi(x) \cdot y) = \varphi^2(x) \cdot \varphi(y) = a \cdot \varphi(b) = a \cdot 0b.$$

Therefore,

$$sx \cdot y = (a \cdot 0b)x \cdot y = (ax \cdot 0b)y = (0 \cdot 0b)y = by = 0,$$

shows that  $s \in A(x, y)$ . Thus the set  $A(x, y)$  is non-empty.

Let  $p$  be an arbitrary element of  $A(x, y)$ . Then  $px$  and  $y$  are in the same branch. Consequently

$$s = 0(0x \cdot y) = 0 \cdot (pp \cdot x)y = 0 \cdot (px \cdot p)y = 0 \cdot (px \cdot y)p = 0 \cdot 0p = \varphi^2(p) \leq p,$$

by Theorem 2.3.

So,  $s$  is the least element of  $A(x, y)$  and  $A(x, y) \subset B(s)$ . □

**Definition 7.3.** We say that a solid weak BCC-algebra  $G$  is *with condition (S)*, if each its subset  $A(x, y)$  has the greatest element. The greatest of  $A(x, y)$  will be denoted by  $x \circ y$ .

**Example 7.4.** Let  $(G; *, 0)$  be an abelian group. Then  $(G; \cdot, 0)$  with the operation  $xy = x*y^{-1}$  is a solid weak BCC-algebra in which each branch has only one element. Thus  $A(x, y) \subset B(x \cdot 0y) = \{x \cdot 0y\}$ . Consequently  $x \circ y = x \cdot 0y = x * y$ . □

**Example 7.5.** Each finite solid weak BCC-algebra decomposed into linearly ordered branches is with condition (S) since each set  $A(x, y)$  is a finite subset of some linearly ordered branch. □

**Example 7.6.** A solid weak BCC-algebra defined by the table:

$\cdot$	0	a	b	c	d
0	0	0	b	b	b
a	a	0	b	b	b
b	b	b	0	0	0
c	c	b	a	0	a
d	d	b	a	a	0

is not with condition (S) since  $A(a, b) = \{b, c, d\} = B(b)$  has no greatest element.  $\square$

Since in a solid weak BCC-algebra  $G$  with condition (S), for each  $x, y \in G$  the set  $A(x, y)$  has the greatest element  $x \circ y$ , so  $\circ$  can be treated as a binary operation defined on  $G$  and  $(G; \circ, 0)$  can be considered as an algebra of type  $(2, 0)$ . Since in any case  $A(x, 0) = A(0, x)$ , the groupoid  $(G; \circ, 0)$  has the identity 0. In the case of BCI-algebras with condition (S),  $(G; \cdot, 0)$  is a commutative semigroup (cf. [18]). For weak BCC-algebras it is not true.

**Example 7.7.** A weak BCC-algebra defined by the table:

$\cdot$	0	1	2	3
0	0	0	2	2
1	1	0	2	2
2	2	2	0	0
3	3	3	1	0

is with condition (S), but in this algebra  $1 \circ 2 \neq 2 \circ 1$  and  $(2 \circ 2) \circ 2 \neq 2 \circ (2 \circ 2)$ .  $\square$

For some BCI-algebras (described in [6] and [7])  $(G; \circ, 0)$  is an abelian group. A similar situation takes place in the case of weak BCC-algebras. To prove this fact we need the following lemma.

**Lemma 7.8.** *In weak BCC-algebras with condition (S)*

$$x \leq y \implies x \circ z \leq y \circ z$$

for all  $x, y, z \in G$ .

*Proof.* If  $x \leq y$ , then also  $(x \circ z)y \leq (x \circ z)x$ , by (3). But, according to the definition,  $(x \circ z)x \leq z$ . Thus  $(x \circ z)y \leq z$ , which implies  $x \circ z \leq y \circ z$  because  $y \circ z$  is the greatest element satisfying the inequality  $py \leq z$ .  $\square$

**Theorem 7.9.** *Let  $(G; \cdot, 0)$  be a weak BCC-algebra with condition (S). Then  $(G; \circ, 0)$  is a group if and only if  $(G; \cdot, 0)$  is group-like.*

*Proof.* Let  $(G; \circ, 0)$  be a group. Consider an arbitrary element  $x \in B(0)$ . Denote by  $x^{-1}$  the inverse element of  $x$  in a group  $(G; \circ, 0)$ . By Lemma 7.8 from  $0 \leq x$  it follows  $x^{-1} = 0 \circ x^{-1} \leq x \circ x^{-1} = 0$ . Hence  $x^{-1} = 0$ . Thus  $B(0) = \{0\}$ . This, by Proposition 3.15 in [12], shows that a weak BCC-algebra  $(G; \cdot, 0)$  is group-like.

Conversely, if a weak BCC-algebra  $(G; \cdot, 0)$  is group-like, then each its branch has only one element. Hence  $px \leq y$  means  $px = y$ , i.e.,  $p * x^{-1} = y$  in the corresponding group  $(G; *, 0)$ . Thus  $p = y * x$  is uniquely determined by  $x, y \in G$ . Therefore  $x \circ y = y * x$ . So,  $(G; \circ, 0)$  is a group.  $\square$

**Corollary 7.10.** *Let  $(G; \cdot, 0)$  be a weak BCC-algebra with condition (S). Then  $(G; \circ, 0)$  is an abelian group if and only if  $(G; \cdot, 0)$  is a group-like BCI-algebra.*

*Proof.* Indeed,  $(G, \circ, 0)$  is abelian if and only if  $(G; \cdot, 0)$  is abelian, that is, if and only if  $xy \cdot z = x * y^{-1} * z^{-1} = x * z^{-1} * y^{-1} = xz \cdot y$ .  $\square$

**Proposition 7.11.** *In a solid weak BCC-algebra with condition (S) we have*

$$(29) \quad xy \cdot z = x(y \circ z)$$

for  $x, y$  belonging to the same branch and  $z \in B(0)$ .

*Proof.* Let  $x, y \in B(a)$ ,  $z \in B(0)$ . Then  $x(xy \cdot z) \cdot y = xy \cdot (xy \cdot z) \leq z$  implies  $x(xy \cdot z) \cdot y \leq z$ . Thus  $x(xy \cdot z) \in A(y, z)$ . Hence  $x(xy \cdot z) \leq y \circ z$  and  $x(y \circ z) \leq x(x(xy \cdot z))$ , by (3). Further, since  $x(xy \cdot z) \in B(a)$ , we have

$$x(x(xy \cdot z)) \cdot (xy \cdot z) = x(xy \cdot z) \cdot x(xy \cdot z) = 0.$$

So,  $x(x(xy \cdot z)) \leq xy \cdot z$ . Consequently,

$$x(y \circ z) \leq x(x(xy \cdot z)) \leq xy \cdot z.$$

On the other hand,  $y \circ z \in A(y, z) \subset B(a)$ , by Proposition 7.2. This, by Lemma 2.4, gives  $x(y \circ z) \in B(0)$ . Thus,

$$(xy \cdot x(y \circ z)) \cdot (y \circ z)y = (xy \cdot (y \circ z)y) \cdot x(y \circ z) \leq x(y \circ z) \cdot x(y \circ z) = 0.$$

Consequently,  $xy \cdot x(y \circ z) \leq (y \circ z)y \leq z$ .

Therefore,

$$0 = (xy \cdot x(y \circ z)) \cdot z = (xy \cdot z) \cdot x(y \circ z),$$

which implies  $(xy \cdot z) \leq x(y \circ z)$ . Hence  $xy \cdot z = x(y \circ z)$ .  $\square$

**Corollary 7.12.** *In a solid weak BCC-algebra with condition (S) the branch  $B(0)$  satisfies the identity (29).*  $\square$

**Corollary 7.13.** *A BCI-algebra with condition (S) satisfies the identity (29).*  $\square$

**Theorem 7.14.** *A solid weak BCC-algebra with condition (S) is restricted if and only if some its branch is restricted.*

*Proof.* Assume that some branch, for example  $B(a)$ , is restricted and  $1_a$  is the greatest element of  $B(a)$ . Then  $xb \in B(0)$  for every  $x \in B(b)$  and an arbitrary  $b \in I(G)$ . Thus  $xb \cdot 0a \in B(0 \cdot 0a) = B(a)$ . Hence  $xb \cdot 0a \leq 1_a$ , i.e.,  $xb \leq 0a \circ 1_a$ , according to the definition of  $0a \circ 1_a$ . Consequently,

$$(30) \quad xb \cdot (0a \circ 1_a) = 0 \quad \text{and} \quad (0a \circ 1_a) \cdot 0a \leq 1_a.$$

Hence  $(0a \circ 1_a) \cdot 0a \in B(a)$ , i.e.,  $a \leq (0a \circ 1_a) \cdot 0a$ . Since  $\varphi(x) = 0x$  is an endomorphism (Proposition 2.11), from the last inequality, applying Theorem 2.3 (2), we obtain

$$0a = 0((0a \circ 1_a) \cdot 0a) = 0(0a \circ 1_a) \cdot (0 \cdot 0a) = 0(0a \circ 1_a) \cdot a.$$

Therefore

$$0 = (0(0a \circ 1_a) \cdot a) \cdot 0a \leq 0(0a \circ 1_a) \cdot 0 = 0(0a \circ 1_a),$$

by (i'). This, by Theorem 2.3 (1), gives  $0 = 0 \cdot 0 = 0 \cdot 0(0a \circ 1_a) \leq 0a \circ 1_a$ . So,  $0a \circ 1_a \in B(0)$ .

Now let  $m = b \circ (0a \circ 1_a)$ . Then for every  $x \in B(b)$ , according to Proposition 7.11 and (30), we have

$$xm = x(b \circ (0a \circ 1_a)) = xb \cdot (0a \circ 1_a) = 0,$$

which implies  $x \leq m$ . Therefore  $m$  is the greatest element of the branch  $B(b)$ .

The converse statement is obvious.  $\square$

## 8. CONCLUSIONS

In the study of various types of algebras inspired by logic a very important role plays the identity  $xy \cdot z = xz \cdot y$  which is not satisfied in weak-BCC-algebras. In this paper we described weak-BCC-algebras satisfying this identity in the case when elements  $x$  and  $y$  (or  $x$  and  $z$ ) are in the same branch. Using the method presented above we can obtain results which are similar to results proved earlier for BCI-algebras. Our method based on the restriction of the verification of various properties to their verification only to elements belonging to the same branch makes it possible to study these properties for the wider class of algebras.

Further results on solid weak BCC-algebras one can find in [20] and [23]. In the first paper some important identities satisfied in weak BCC-algebras are described; in the second –  $f$ -derivations of weak BCC-algebras. Since obtained results are very similar to those proved for example for BCI-algebras seems that verification of various properties can be reduced to verification in branches only which is very important for computer verification.

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